CAUSALITY, SHOCKS AND INSTABILITIES IN VECTOR FIELD MODELS OF LORENTZ SYMMETRY BREAKING.

M. A. CLAYTON

Abstract. We show that that vector field-based models of the ether generically do not have a Hamiltonian that is bounded from below in a flat spacetime. We also demonstrate that these models possess multiple light cones in flat or curved spacetime, and that the non-lightlike characteristic is associated with an ether degree of freedom that will tend to form shocks. Since the field equations (and propagation speed) of this mode is singular when the timelike component of the ether vector field vanishes, we demonstrate that linearized analyses about such configurations cannot be trusted to produce robust approximations to the theory.

1. Introduction

Recently there has been interest in models in which constants of nature are elevated to the status of fields [1, 2, 3]. This is partly due to recent experimental evidence for a variation of the fine structure constant with time [4, 5, 6, 7], and partly from theoretical interest in, for example, solving the horizon and flatness problems in cosmology [8, 1] or modeling Planck scale deviations from flat spacetime field theory in the very early universe [9].

Along with these models, either explicitly or implicitly, comes a breakdown of Lorentz invariance. This appears as contributions to the action that are explicitly written in a preferred frame [10], or in a more subtle manner where, although no preferred frame is chosen, one or more of the equivalence principles are violated [11, 12, 3]. Indeed, the motivation for these models is to embody the idea of a "varying speed of light", which should conflict with Einstein's postulate of the invariance of the propagation velocity of light.

Among the latter class of models we would include those that have similar prior-geometric structure [13], the so-called "k-inflation" and "k-essence" models [14, 15], and vector field models of the ether [16, 17, 18]. Although these models are not necessarily intended introduce fields with dynamically determined light velocities, this nevertheless results from their construction. In the same way, higher-dimensional models [19, 20, 21] also generically end up with multiple light cones as a result of dimensional reduction.

The class of model that we want to discuss here are those considered, for example, in [16, 17, 18]. These models use a (co-)vector field A_μ to model the preferred frame representing the ether, introducing a constraint in the action via a Lagrange multiplier λ that forces A_μ to be a timelike, null or spacelike vector. The model is

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described by the action:

$$S = -\int d\mu R + \int d\mu\Big(-\frac{1}{4}F^2 + \frac{1}{2}\lambda(A^2 - l) + A_\mu J^\mu\Big), \tag{1}$$

where we are using a metric with signature $(+,-,-,-)$, have chosen $16\pi G = 1 = c$, and have normalized the vector field A_μ to remove a possible arbitrary coefficient multiplying the F^2 term. We have also written $d\mu = \sqrt{-\mathrm{g}}d^4x$, $A^2 = \mathrm{g}^{\mu\nu}A_\mu A_\nu$, and $F^2 = \mathrm{g}^{\mu\alpha}\mathrm{g}^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$. The source-coupling term $A_\mu J^\mu$ is included to represent the possible non-derivative coupling of the ether to matter fields through a matter action: $S_{\mathrm{matter}}(\mathrm{g}, A, \dots)$, and so one may think of $J^\mu = \partial S_{\mathrm{matter}}/\partial A_\mu$.

There is a fairly long history of difficulties associated with coupling vector fields to gravity. Kuchař [22] has noted that in general, fields that are derivatively-coupled to gravity may propagate off the light cone as determined by the spacetime metric. Examples of this happening when 'higher spin' fields were coupled to gravity have been in existence for quite some time [23, 24, 25, 26], and linear models coupling a vector field to gravity are known to possess instabilities unless they are one of Maxwell, Proca, or a purely longitudinal vector field, minimally-coupled to gravity [27]. In this paper we demonstrate that the models described by (1) suffer from similar problems, not so much to point out flaws in the literature as to further illustrate the consequences of modifying the structure of vector field models. As these issues are serious, they play a complimentary role to observational constraints arising from, for example, performing a PPN expansion (see for example [28, 29]).

In Section 2, we begin an examination of the model on a flat spacetime, and show that the constraint imposed by λ re-introduces the spatially longitudinal mode of the vector field as a physical field, giving it a propagation speed that depends nonlinearly on the vector field itself. We explicitly determine this velocity, and show that for spacelike ether models it is infinite near configurations with $A_0 = 0$. In Section 3 we show that the linearization of these models does not correctly reproduce this result, and in Section 4, we demonstrate that the field equations are also singular at this point. Also in this section, we show that the the field equations for timelike or spacelike ether models, will tend to form shocks.

We also show, in Section 5, that these models generically do not have a classical energy that is bounded from below regardless of the signature of the ether, and in Section 6 we briefly consider the implications of adding additional terms to the action, and argue that such terms are not likely to help matters. Rather than proceed to a covariant or Hamiltonian analysis of the model in a curved spacetime, in Section 7 we consider a plane-symmetric spacetime, and reproduce the same propagation velocities as were observed in a flat spacetime. We conclude in Section 8 with some overall comments.

2. Flat Spacetime

We begin by considering the model (1) in flat spacetime, since it is vastly simpler to illustrate some important dynamical aspects of the model without dealing with gravitational interactions. That we can do so and obtain results that are applicable to curved spacetimes is not a trivial issue though, as we shall describe more fully in Section 6. In any case, this sort of flat spacetime approximation (that is, so that one can transform $\mathrm{g}_{\mu\nu} \approx \eta_{\mu\nu}$ in a region of spacetime and consider small perturbations about this background) should be reasonable if $\mathrm{g}_{\mu\nu}$ is in some sense the physically meaningful metric of spacetime.

The Lagrange multiplier field λ enforces the constraint:

$$A^2 = A_0^2 - \vec{A}^2 = l, \tag{2}$$

where the sign of l indicates whether the vector field A_μ describes a timelike ($l > 0$), null ($l = 0$) or spacelike ($l < 0$) ether. Here we are writing the spacetime covector field in terms of a scalar and spatial vector as: $A_\mu = (A_0, \vec{A})$, and will use, for example $\vec{A} \cdot \vec{B} = \sum_i A_i B_i$ and $\vec{\partial}^2 = \sum_i \partial_i^2$. Variation of (1) with respect to A_μ leads to the (nonlinear) field equations:

$$\partial_\nu F^{\mu\nu} - \lambda A^\mu = J^\mu. \tag{3a}$$

Contraction of this with respect to A^μ and using (2) yields

$$l\lambda = A_\mu \partial_\nu F^{\mu\nu} - A_\mu J^\mu, \tag{3b}$$

which determines λ provided that $l \neq 0$. Taking the divergence of (3a) yields the constraint

$$\partial_\mu[\lambda A^\mu] = -\partial_\mu J^\mu. \tag{3c}$$

The first interesting feature of these models is that they possess multiple light cones, or multiple characteristics, indicating that there are fields in the theory that propagate at different speeds. Since this is not obvious by cursory examination of (3a), we will explicitly determine these propagation velocities.

2.1. **The Reduced Timelike and Null Field Equations.** In both of these cases ($l > 0$ and $l = 0$ respectively) we can use (2) to determine A_0 since the only way that A_0 can vanish is if A_μ vanishes identically.

In the timelike case (3b) will determine λ, and the three remaining equations in (3a) should determine $\vec{A}$. To examine these equations we introduce a vector s_μ such that $s_\mu A^\mu = 0$, which is a four vector $s_\mu = (s_0, \vec{s})$ with arbitrary spatial components s_i, and timelike component determined from

$$s_0 = \vec{n} \cdot \vec{s}, \tag{4}$$

where we use the definition

$$\vec{n} = \frac{\vec{A}}{A_0}. \tag{5}$$

Summing (3a) into s_μ and using (4) to write what results in terms of s_i gives

$$s_\mu \partial_\nu F^{\mu\nu} = s_i \left[n_i \partial_\alpha F^{0\alpha} + \partial_\alpha F^{i\alpha} \right] = s_i (n^i J^0 + J^i). \tag{6}$$

Since s_i is arbitrary, the vector in square brackets is the three field equations that we are seeking:

$$n_i \partial_j F_{0j} + \partial_t F_{i0} - \partial_j F_{ij} = -(n^i J^0 + J^i). \tag{7}$$

The same equation is arrived at in the null case by solving the zeroth component of (3a) for λ (this is a true algebraic equation for λ since it involves no second-order time derivatives of any field) and substituting it into the spacelike equations.

Re-written in terms of the vector potential, (7) becomes

$$\partial_t^2 A_i - \partial_t \partial_i A_0 - n_i \partial_t \partial_j A_j - \vec{\partial}^2 A_i + n_i \vec{\partial}^2 A_0 + \partial_i \partial_j A_j = n_i J^0 + J^i. \tag{8}$$

These equations can be written solely in terms of $\vec{A}$ using the derivative relations that follow from (2):

$$\partial_t A_0 = \vec{n}\cdot\partial_t\vec{A}, \quad \partial_i A_0 = \vec{n}\cdot\partial_i\vec{A}, \tag{9}$$

which leads to a second-order, quasi-linear partial differential equation of the form:

$$L_{ij}(A_k, \partial_t^2, \partial_t\partial_k, \partial_k\partial_l)A_j + b_i(A_k, \partial_t A_k, \partial_l A_k) = 0, \tag{10a}$$

where

$$L_{ij} = \big[\delta_{ij}\partial_t^2 - n_j\partial_t\partial_i - n_i\partial_t\partial_j - \delta_{ij}\vec{\partial}^2 + n_in_j\vec{\partial}^2 + \partial_i\partial_j\big], \tag{10b}$$

$$b_i = A_0^{-1}n_i(\delta_{kl} - n_kn_l)\partial_jA_k\partial_jA_l - A_0^{-1}(\delta_{jk} - n_jn_k)\partial_iA_j\partial_tA_k. \tag{10c}$$

While the lower-order derivative terms (10c) are important to determine the particular form of the solutions of (10), in this case they are irrelevant when determining the characteristics (see, for example, the discussion in [30]). For this second-order, quasilinear system, the characteristic polynomial (see in particular the discussion in Section VI.6 in [31]) may be found by making the replacement $\partial_i \to ip_i$ and $\partial_t \to ip_0$ in L_{ij}, and computing the determinant

$$\det(L_{ij}) = (p_0^2 - \vec{p}^2)^2(p_0 - \vec{n}\cdot\vec{p})^2 = 0. \tag{11}$$

(This is most easily accomplished performing a rotation so that $\vec{n} = (n_1, 0, 0)$ which we can do since we are working pointwise–the results can then be re-assembled to give the general result.) This shows us that the initial value problem is well-posed provided we are imposing initial data on non-characteristic surfaces, where is this case the condition for a hypersurface to be non-characteristic depends on the vector field A_i. This is not particularly unusual, as exactly this situation also occurs in general relativity where the condition that data be imposed on a spacelike hypersurface involves the metric tensor. While (10) is not diagonal, and so the discussions in, for example [32, 33], do not apply directly, it may be made so following the procedure given in [31].

We see therefore that there are characteristics defined by

$$p_0 = \pm|\vec{p}|, \quad \text{and} \quad p_0 = \vec{n}\cdot\vec{p}, \tag{12a}$$

two of which are the usual null propagation conditions, and the third depends on nonlinear contributions from $\vec{A}$. We therefore see quite clearly why this mode was not so well understood when linearizing about a flat background with $\vec{A} = 0$ [16]: in the linearized limit we would have a $p_0 = 0$ ‘characteristic’, which could be interpreted as a non-propagating field (that this mode should propagate was recognized in [16]).

Before considering the spacelike case, we note that we can write (12a) in the covariant form:

$$p^2 = 0, \quad \text{and} \quad A_\mu p^\mu = 0, \tag{12b}$$

and so it is not too surprising that we will find exactly the same result. As we will see in Section 4, and later on in a curved spacetime in Section 7, the field equations associated with the new characteristic surfaces have a tendency to form shocks. Note that although we have performed this analysis in three spatial dimensions, the result should hold in any number of spacetime dimensions.

2.2. **The Reduced Spacelike Field Equations.** It requires a bit more effort to show that (12) also holds in the spacelike case. To begin, we write $A_\mu = (A_0, A\hat{a})$ where $\hat{a}$ is a unit vector in the direction of $\vec{A}$ and A is the magnitude of the spatial component (so $A = \sqrt{\vec{A}^2}$ and $\hat{a} = \vec{A}/A$). The normalization condition is $A_0^2 - A^2 = l < 0$, which can be solved for A (this requires that $A > 0$, and it is now possible that $A_0 = 0$).

We again introduce a vector s_μ satisfying $s_\mu A^\mu = 0$, but in this case we have $s_0 A_0 = A\vec{s}\cdot\hat{a}$, and therefore the vector $\vec{s}$ must have the form

$$\vec{s} = s_A \hat{a} + \vec{s}_T, \quad \text{where} \quad \vec{s}_T \cdot \hat{a} = 0. \tag{13}$$

Again summing s_μ into (3a) gives only three equations, one proportional to s_0 and the other two proportional to $\vec{s}_T$:

$$s_0\Big(\partial_\nu F^{0\nu} + \frac{A_0}{A} a_i \partial_\nu F^{i\nu}\Big) + s_{Ti}\partial_\nu F^{i\nu} = s_0\Big(J^0 + \frac{A_0}{A} a_i J^i\Big) + s_{Ti} J^i, \tag{14}$$

which can be written as

$$\partial_\nu F^{0\nu} + \frac{A_0}{A} a_j \partial_\nu F^{j\nu} = J^0 + \frac{A_0}{A} a_j J^j, \tag{15a}$$

$$(\delta_{ij} - a_i a_j)\partial_\nu F^{j\nu} = (\delta_{ij} - a_i a_j) J^j. \tag{15b}$$

Multiplying the second by A_0/A and adding the first multiplied by a_i, we have

$$a_i \partial_\nu F^{0\nu} + \frac{A_0}{A}\partial_\nu F^{i\nu} = a_i J^0 + \frac{A_0}{A} J^i, \tag{16}$$

which are three equations that correctly reproduce (15).

Since (16) is equivalent to (7) up to a multiplicative factor A_0/A, the analysis that we performed to find the characteristic surfaces will proceed as before, and lead to (12). The analysis has properly taken care of the fact that here $\vec{A}$ cannot vanish and A_0 can, and we see from (12) that in the $A_0 \to 0$ limit the speed of propagation of one of the modes goes to infinity.

3. Perturbative Analysis

We consider a generic background A_μ and perform a linearized expansion: $A_\mu \to A_\mu = \delta A_\mu$, $\lambda = \lambda + \delta\lambda$ and $J^\mu \to J^\mu + \delta J^\mu$. From the expansion of (2) we see that

$$A^\mu \delta A_\mu = 0, \tag{17a}$$

regardless of the sign of l. Then *provided that* $A_0 \neq 0$ we can proceed in a similar manner as in Section 2, which leads to the perturbation equations

$$\partial_j \delta F_{ij} - \partial_t \delta F_{i0} - n_i \partial_j \delta F_{0j} + \lambda(\delta_{ij} - n_i n_j)\delta A_j = \delta J^i + n_i \delta J^0. \tag{17b}$$

From this we see that this has a principle part that is equivalent to (7), and therefore perturbations will have characteristics determined by (12). This result is redundant given that we had already determined the characteristics in the previous section, but may be looked upon as an alternative derivation.

That (17b) is correct unless $A_0 = 0$, should be a signal that there is something special about such configurations. What we are going to show is that performing a linearization about such background solutions does not produce linearized equations that approximate exact solutions of the field equations. This is a Cauchy instability [33] in the exact theory, which shows up as a linearization instability in the perturbative calculation (see, for example [34, 35]). In this section we will show

that the linearized equations only have $p^2 = 0$ characteristics, contradicting the exact results (12). We will also show in Sections 4, 5 and 7 that the field equations are singular in the $A_0 \to 0$ limit.

We must first consider background solutions with $A_0 = 0$. Note that in order to write the field equations in the form (16) we multiplied (15b) by A_0, and therefore taking $A_0 = 0$ in (16) will not result in a full set of field equations. From (2) we must have $\vec{A}^2 = -l > 0$, so we see that we only have two degrees of freedom remaining. The zeroth component of (3a) results in

$$\partial_t \partial_i A_i = -J^0, \tag{18}$$

and (3b) determines λ as:

$$l\lambda = A_i(\partial_i \partial_j A_j - \vec{\partial}^2 A_i + \partial_t^2 A_i) - A_i J^i. \tag{19}$$

Since this determines λ we can use this to write the remaining two spatial components of (3a) as

$$P_{ij}(\partial_t^2 A_j + \partial_j \partial_k A_k - \vec{\partial}^2 A_j) = P_{ij} J^j, \tag{20a}$$

where the projection tensor is defined as

$$P_{ij} = \delta_{ij} + l^{-1} A_i A_j. \tag{20b}$$

Note that we have three equations for the two remaining functions in A_i. This is not surprising since we have imposed $A_0 = 0$ by hand, and we expect that this will only be possible with some restriction on the source J^μ. Rather than determining the general form of these restrictions, we will choose a particular solution with: $J^0 = 0$ and $J^i = 0$, which implies that $\lambda = 0$ and allows a constant A_i.

Performing the linearized expansion about this solution, from (17a) we find that $A_i \delta A_i = 0$. The timelike component of the linearization of (3a) leads to

$$\partial_i^2 \delta A_0 - \partial_t \partial_i \delta A_i = \delta J^0, \tag{21a}$$

and summing A_i into the linearization of the spacelike components of (17a) yields an equation for $\delta\lambda$:

$$l\delta\lambda = -A_i\big(\partial_t \delta F_{i0} - \partial_j \delta F_{ij}\big) - A_i \delta J^i. \tag{21b}$$

Using this to remove $\delta\lambda$ from the remaining perturbation equations results in

$$P_{ij}\big(\partial_t \delta F_{i0} - \partial_j \delta F_{ij}\big) = -P_{ij} \delta J^j. \tag{21c}$$

Assuming that A_i is a constant vector pointing along the x-direction: $A_i = (a, 0, 0)$ where $a^2 = -l$, then (17a) implies that $\delta A_x = 0$. It is then a fairly straightforward matter to re-write the field equations that result from (21) as:

$$(\partial_t^2 - \vec{\partial}^2)\partial_x^2 \delta A_0 = (\partial_t^2 - \partial_x^2)\delta J^0 + \partial_t(\partial_y \delta J^y + \partial_z \delta J^z), \tag{22a}$$

$$(\partial_t^2 - \vec{\partial}^2)\partial_x^2(\partial_y \delta A_y + \partial_z \delta A_z) = \partial_x^2(\partial_y \delta J^y + \partial_z \delta J^z) + \partial_t(\partial_y^2 + \partial_z^2)\delta J^0, \tag{22b}$$

$$(\partial_t^2 - \vec{\partial}^2)(\partial_y \delta A_z - \partial_z \delta A_y) = \partial_z \delta J^y - \partial_y \delta J^z. \tag{22c}$$

This system contains three modes that all propagate at the speed of light, with no sign of the mode that propagates along the $p_\mu A^\mu = 0$ characteristic.

These linearized equations would therefore lead one to believe that small fluctuations about a field configuration with $A_0 = 0$ *all* propagate at the speed of light. We know from the exact field equations that this is most definitely *not* the case,

and that there is a degree of freedom that has an infinitely large propagation velocity in the limit that $A_0 = 0$. It should be reasonably clear that this is happening because we are attempting to perform a perturbation analysis about a degenerate field configuration–in fact our initial data surface is a characteristic surface for the exact field equations.

This suggests us that the expansion performed in [18] is suspect. It is, of course, possible that other models will result in the linearized equations that they use, but it is clear from what we have seen here that the linearized equations that they derive do *not* approximate solutions to the exact field equations. Lest one think that this sort of thing would not happen when the model is coupled to gravity, in Section 7 we show that the same characteristics (12) appear in a curved spacetime.

We also see that this does not happen in the null and timelike cases, but this does not mean that there are not other issues to deal with. We will show in the next section that shock formation should be expected, and in Section 5, that the (classical) Hamiltonian of these models will not be bounded from below.

4. Shock Formation

Now we wish to examine some properties of nontrivial solutions that will display propagation along the $p_\mu A^\mu = 0$ characteristic. One such example is that of a plane-symmetric vector potential: $A_\mu = (A_0(t,x), A_1(t,x), 0, 0)$ (and correspondingly for the source), since these solutions are expected to propagate along the x-direction, along which $\vec{A}$ also points. What we will discover (for a non-null ether) is that the field equations reduce to a form that can be written as a conservation law and is generically expected to form shocks.

From (3a) we find the two field equations:

$$\partial_x^2 A_0 - \partial_t \partial_x A_1 - \lambda A_0 = J^0, \tag{23a}$$

$$\partial_t^2 A_1 - \partial_t \partial_x A_0 + \lambda A_1 = J^1, \tag{23b}$$

and taking a linear combination that removes λ we find:

$$A_1\big(\partial_x^2 A_0 - \partial_t \partial_x A_1\big) + A_0\big(\partial_t^2 A_1 - \partial_t \partial_x A_0\big) = A_1 J^0 + A_0 J^1. \tag{24}$$

Using (2), this can be written as an equation determining either A_0 or A_1, both forms of which are:

$$\Big(\partial_t - \frac{A_1}{A_0}\partial_x\Big)\Big(\partial_t A_1 - \frac{A_1}{A_0}\partial_x A_1\Big) = \frac{A_1}{A_0}J^0 + J^1, \tag{25a}$$

$$\Big(\frac{A_0}{A_1}\partial_t - \partial_x\Big)\Big(\frac{A_0}{A_1}\partial_t A_0 - \partial_x A_0\Big) = J^0 + \frac{A_0}{A_1}J^1. \tag{25b}$$

In either case we see that in a region external to the sources we need only consider a homogeneous, first-order, partial differential equation, solutions to which are also solutions when the source is non-vanishing.

For a null ether we have $A_0 = \pm A_1$, and (25a) leads to

$$\partial_t A_1 \mp \partial_x A_1 = 0, \tag{26a}$$

with solutions: $A_1 = A_1(t \pm x)$. Throughout we will assume that $A_0 > 0$ (the ether is forward-pointing in time), and so disturbances in the ether field propagate in the opposite spatial direction to which the ether field itself points. This last fact is a generic feature, consistent with the characteristic surface defined by $p_\mu A^\mu = 0$ in (12).

For a timelike either we can solve (2) for A_0 as: $A_0 = \pm\sqrt{A_1^2 + l}$, and so we consider solutions of:

$$\partial_t A_1 \mp \frac{A_1}{\sqrt{A_1^2 + l}} \partial_x A_1 = \partial_t A_1 \mp \partial_x \sqrt{A_1^2 + l} = 0, \tag{26b}$$

where the second form is written as a conservation law. This type of equation is considered in many standard texts on fluid or gas dynamics (see for example [30, 36]), from which it is evident that, for example, a Gaussian fluctuation of A_1 about a constant background A_0 will inevitably lead to shock formation. The value of A_1 is constant along the characteristics, which are straight lines determined from the initial value of A_1 by: $x(t) = k(x_0)t + x_0$ where $k(x_0) = \mp A_1(x_0)/\sqrt{A_1^2(x_0) + l}$ and $A_1(x_0) = A_1(0, x_0)$. Note though that since $l > 0$ we have $|k| < 1$, and disturbances propagate at a velocity slower than that of light.

For a spacelike either we have a similar situation. Solving (2) for A_1 and using (25b) to determine A_0, we find (this can also be written as a conservation law):

$$\partial_t A_0 \mp \frac{\sqrt{A_0^2 - l}}{A_0} \partial_x A_0 = 0, \tag{26c}$$

which shows that disturbances not only generically form shocks but that they also travel faster than light. We also see quite clearly that the limit $A_0 \to 0$ does not lead to a smooth linear limit. Writing the characteristics as above, we now have $k(x_0) = \mp\sqrt{A_0^2 - l}/A_0$, and regions with very small A_0 have characteristics that approach infinite velocity. Since disturbances in A_0 cannot be assumed to be exactly constant over an extremely large region, shock formation is to be expected.

Although we have explored the system in a plane-symmetric situation (which we shall explore further in Section 7 in a curved spacetime), we have found that an identical result holds for spherically-symmetric solutions. Although a spherically-symmetric ether is difficult to motivate, it does show that the existence of shocks is not an artifact of the specific solution that we are considering.

At this point it is worthwhile saying a few words about a restricted gauge invariance that the system possesses when the external source coupling is conserved (that is, when $\partial_\mu J^\mu = 0$). In this case the action (1) is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \theta$ provided that θ satisfies: $\mathrm{g}^{\mu\nu}(A_\mu + \partial_\mu\theta)(A_\nu + \partial_\nu\theta) = l$. In the present model, this simplifies to (solving for $\partial_t\theta$):

$$\partial_t \theta = -A_0 \pm \sqrt{A_0^2 + \partial_x\theta(\partial_x\theta + 2A_1)}. \tag{27}$$

It can be shown that all of (26) are invariant under this transformation, so that A_μ may be transformed to zero by an appropriate choice of gauge.

This should not be too surprising, since in this case the model is identical to a Maxwell field in the nonlinear gauge: $A^2 = l$. Therefore, if the ether field is only coupled to a conserved current then the model is gauge-equivalent to a Maxwell field, and there would be little point working in the nonlinear gauge (2) when some more well-behaved linear gauge (such as the Lorenz gauge $\partial_\mu A^\mu = 0$) would equivalent and far be simpler to work with. The intended interpretation of the model is that A_μ should represent an ether field, and therefore it will couple to matter fields as a timelike metric representing the preferred frame: $\mathrm{g}^{\mathrm{pf}}_{\mu\nu} \sim A_\mu A_\nu$ [16, 17], and so coupling to a conserved current is neither expected nor desired. For the

ether to have observable consequences it must couple to a non-conserved current, in which case the $p_\mu A^\mu = 0$ characteristic is physical, and shock formation is expected.

5. Canonical Analysis

The existence of multiple light cones and shock formation in a theory of the ether is perhaps not sufficient reason to discard the model. The theories that we are considering here have another dangerous feature though: they do not have a classical Hamiltonian that is bounded from below in flat spacetime. This analysis should be enough to convince the reader that in any (stable) region of spacetime that can be approximated by a flat spacetime, these theories should be generically unstable once quantum or other dissipative effects are included.

In a quantum scenario, the standard argument is that quantum fluctuations can result in the creation of real (as opposed to virtual), negative energy particles, since no energy need be 'borrowed' from the vacuum to create them. Thus the quantum system will cascade into states of increasingly more negative energy. While it is possible to claim that these negative energy states are not realized in a classical model (if the initial data has finite energy it will remain with the same finite energy), vacuum fluctuations in the quantum theory seem unavoidable.

The analysis is fairly standard, so we will be brief. Expanding the Lagrangian from (1) we find

$$L = \int d^3x \left[\frac{1}{2}(\partial_t A_i - \partial_i A_0)^2 - \frac{1}{4}(F_{ij})^2 + \frac{1}{2}\lambda(A_0^2 - \vec{A}^2 - l) + A_0 J^0 + \vec{A}\cdot\vec{J}\right], \tag{28}$$

and we see that we can treat A_0 and λ as Lagrange multipliers, and the conjugate momentum for $\vec{A}$ is

$$P_i = \frac{\delta L}{\delta \partial_t A_i} = \partial_t A_i - \partial_i A_0. \tag{29}$$

The Hamiltonian is then

$$\begin{aligned} H &= \int d^3x P_i \partial_t A_i - L \\ &= \int d^3x \left[\frac{1}{2}\vec{P}^2 + \vec{P}\cdot\vec{\nabla}A_0 + \frac{1}{4}(F_{ij})^2 - \frac{1}{2}\lambda(A_0^2 - \vec{A}^2 - l) - A_0 J^0 - \vec{A}\cdot\vec{J}\right], \end{aligned} \tag{30}$$

and we will assume that the fields fall off fast enough as $r \to \infty$ that no surface contributions are required.

The condition imposed by λ is (2), which can be used to determine the Lagrange multiplier A_0 as

$$A_0 \approx \pm\sqrt{\vec{A}^2 + l}. \tag{31}$$

This is a delicate point since, as noted before, if $l < 0$ there is no guarantee that $A_0 \neq 0$, but since (2) involves the the Lagrange multiplier A_0 it cannot be considered as a constraint. The condition imposed by A_0 is

$$\frac{\delta H}{\delta A_0} = -\vec{\nabla}\cdot\vec{P} - \lambda A_0 - J^0 \approx 0, \tag{32}$$

so that since A_0 is already determined, this determines the Lagrange multiplier λ as:

$$\lambda \approx \mp \frac{\vec{\nabla}\cdot\vec{P} + J^0}{\sqrt{\vec{A}^2 + l}}. \tag{33}$$

Evaluating the Hamiltonian (30) "on-shell" (replacing the Lagrange multipliers by their phase space representation using (31) and (33)) in a region where the source terms vanish, we find the remaining contributions:

$$H \approx \int d^3x \Big[\frac{1}{2}\vec{P}^2 \pm \frac{P_iA_j\nabla_iA_j}{\sqrt{\vec{A}^2 + l}} + \frac{1}{4}(F_{ij})^2\Big], \tag{34}$$

where we have used $\partial_i A_0 = \pm(A_j\partial_iA_j)/\sqrt{\vec{A}^2+l}$. To show that this can be arbitrarily negative on phase space, consider configurations derivable from a static scalar potential as $A_i = \partial_i\phi(\vec{x})$. The conjugate momentum to $\vec{A}$ is $P_i = -\partial_iA_0$, and evaluating the Hamiltonian density yields: $-(1/2)(\partial_iA_0)^2$, which can be made arbitrarily large and negative by choosing ϕ to be a Gaussian with large amplitude. Note how this has occurred: the imposition of (2) has caused (32), which would have been the Gauss constraint in Maxwell's theory and led to a positive-definite term in the Hamiltonian, to now become an equation that determines a Lagrange multiplier, and the Hamiltonian is indefinite.

We can also consider Hamilton's equations, which appear fairly innocent when written in terms of the Lagrange multipliers:

$$\dot{A}_i = \{A_i, H\} = \frac{\delta H}{\delta P_i} = P_i + \partial_iA_0 \approx P_i \pm \frac{A_j\partial_iA_j}{\sqrt{\vec{A}^2+l}}, \tag{35a}$$

$$\begin{aligned}\dot{P}_i = \{P_i, H\} = -\frac{\delta H}{\delta A_i} &= -\partial_jF_{ij} - \lambda A_i + J^i \\ &\approx -\partial_jF_{ij} \pm A_i\frac{\vec{\nabla}\cdot\vec{P} + J^0}{\sqrt{\vec{A}^2+l}} + J^i,\end{aligned} \tag{35b}$$

where in the second form of these (31) and (33) have been imposed. Note that as far as the initial value formulation is concerned the sign of l is largely irrelevant, but manifests itself by allowing the possibility that $A_0 = 0$ in the spacelike case, in which limit (35b) is singular. This is another reflection of the fact that in this model $A_0 = 0$ is a special point, and we cannot expect that it may be reached smoothly (see also [27]).

6. Related Models

Before demonstrating that all of the aforementioned issues also arise in the curved spacetime form of the model, we make a few comments on flat spacetime generalization of (1).

If we were to introduce non-Maxwell kinetic contributions such as $(\nabla_\mu A^\mu)^2$ into (1) as suggested in [17], we would perhaps have different characteristics, but all of the difficulties that we have been discussing here would be present. Since the model will have time derivatives of A_0 in the action, A_0 is no longer a Lagrange multiplier and the $\vec{P}\cdot\vec{\partial}A_0$ term remains in the Hamiltonian, along with new contributions $P_0^2/(2d) + P_0\vec{\partial}\cdot\vec{A}$ involving P_0 (the momentum conjugate to A_0). In this case since both P_0 and A_0 are phase space coordinates, and it is a trivial matter to find field configurations on which the Hamiltonian is as large and negative as

one likes. In order for this theory to be equivalent to a Maxwell field written in a nonlinear gauge one would need a restricted gauge invariance that simultaneously left the $(\nabla_\mu A^\mu)^2$ term and the condition (2) invariant, which seems unlikely. That this type of model is 'dangerous' in this sense has already been discussed in the literature [27]. The same reference also shows that the model considered as an example of a preferred-frame theory of gravity in [28], has stability issues that go beyond the PPN analysis performed there.

Similar comments apply to the possible addition of curvature-coupling terms such as $R_{\mu\nu}A^\mu A^\nu$. Since this would constitute an explicit derivative-coupling, altered characteristics are to be expected [22], and were observed in [37]. In the latter reference the characteristic surfaces of the exact theory are not determined, and so it is not known whether the perturbation that is performed is degenerate in the sense discussed in Section 3.

If we alter the term that imposes the constraint (2) to $\lambda(A^2 - l)^2$ as suggested in [18], then λ will drop out of the field equations altogether, leaving (2) and $\partial_\nu F^{\mu\nu} = J^\mu$. Consistency of the field equations *requires* that the vector source is conserved: $\partial_\mu J^\mu = 0$, and since the divergence equation is trivial there are really only four field equations. Since the coupling is therefore required to be gauge-invariant, this is a Maxwell-theory written in a nonlinear gauge defined by (2). Therefore the $n = 2$ model of [18] is just Einstein-Maxwell theory in disguise, with the Lorentz symmetry breaking as the result of a nonlinear gauge choice for the vector field. It is interesting that adding a self-coupling potential term of the form $V(A^2)$ to this theory could have disastrous consequences, since even if the coupling current is conserved the divergence of the field equations lead to $\partial_\mu A^\mu = 0$, which is a nontrivial equation determining a degree of freedom in A_μ, therefore there are five field equations for the four unknown functions A_μ. In contrast, introducing a self-interaction term to (1) merely induces the shift: $\lambda \to \lambda + V'(l)$. This demonstrates that the inclusion (or not) of a mass term in a vector field theory can have rather significant consequences, contrary to the statement made in [37, Section 5.4].

We should mention that the vector field-based model that we have introduced [11] is of a different class than those discussed here. Because we are imposing no additional constraints in addition to those already present in an Einstein-Proca model, we do not expect the issues discussed in this paper to arise. On the other hand, since the vector field is not guaranteed to be timelike in these models (although this turns out to be the case in a cosmological setting), it is not so obvious that we can interpret the vector field as representing a dynamical ether fluid in general.

7. Plane-Symmetric Spacetimes

We will now examine the model in a particular curved spacetime, demonstrating that the characteristics (12), existence of shocks, and problems performing a linear expansion near $A_0 = 0$ all survive. This is a curved spacetime generalization of the plane-symmetric solutions considered in Section 7, which is general enough to contain the static background solutions that were considered in [18]. We can therefore examine some nonlinear solutions 'near' this spacetime.

Working now in a curved spacetime, from (1) we find the field equations for A_μ:

$$\mathrm{g}^{\mu\nu} A_\mu A_\nu = l, \quad \nabla_\nu F^{\mu\nu} - \lambda A^\mu = 0, \tag{36a}$$

and the Einstein equations:

$$G_{\mu\nu} = -\frac{1}{2}\Big(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}\mathrm{g}_{\mu\nu}F^2\Big) + \frac{1}{2}\lambda A_\mu A_\nu. \tag{36b}$$

The spacetimes we are interested in have an ‘ether wind’ along one direction:

$$A_\mu = (A_0(t,x), A_1(t,x), 0, 0, \dots, 0), \tag{37}$$

consistent with translation and rotation killing vectors in a flat n-dimensional hypersurface, with plane-symmetric metric:

$$\mathrm{g}_{\mu\nu} = \begin{pmatrix} \mathrm{g}_{ab}(t,x) & 0 \\ 0 & -\alpha^{4/n}(t,x)\delta_{ij} \end{pmatrix}, \tag{38}$$

where $a, b, \ldots \in \{1, 2\}$ and $i, j, \ldots \in \{3, 4, \ldots, n+2\}$, so that n is the dimension of the flat subspace. We use coordinates $\{y^i\}$ in the flat hypersurface and t and x in the ‘curved’ directions. We will also use the remaining coordinate invariance in the $t - x$ plane to choose the conformally-flat metric

$$\mathrm{g}_{ab} = e^{\phi}\eta_{ab}, \tag{39}$$

where η_{ab} is the two-dimensional Minkowski metric: $\eta_{ab} = \mathrm{diag}(1, -1)$.

Reducing the action results in (dividing out the infinite volume $\int d^n y$ in the flat plane):

$$\begin{aligned} S_{\mathrm{red}} = \int dt\, dx\, \Big[-2\alpha\eta^{ab}\partial_a\alpha\partial_b\phi - \frac{4(n-2)}{n}\eta^{ab}\partial_a\alpha\partial_b\alpha \\ - \frac{1}{4}\alpha^2 e^{-\phi}F_{01}^2 + \frac{1}{2}\lambda\alpha^2(\eta^{ab}A_aA_b - le^{\phi})\Big], \end{aligned} \tag{40}$$

where

$$F_{01} = \partial_t A_1 - \partial_x A_0. \tag{41}$$

From this we find the field equations

$$(\partial_t^2 - \partial_x^2)\alpha^2 + \alpha^2\Big(\frac{1}{4}e^{-\phi}F_{01}^2 - \frac{1}{2}l\lambda e^{\phi}\Big) = 0, \tag{42a}$$

$$(\partial_t^2 - \partial_x^2)\phi + \frac{4(n-2)}{n}\frac{1}{\alpha}(\partial_t^2 - \partial_x^2)\alpha - \frac{1}{2}e^{-\phi}F_{01}^2 = 0, \tag{42b}$$

$$A_0^2 - A_1^2 - le^{\phi} = 0, \tag{42c}$$

$$\partial_x\big[\alpha^2 e^{-\phi}F_{01}\big] - 2\lambda\alpha^2 A_0 = 0, \tag{42d}$$

$$\partial_t\big[\alpha^2 e^{-\phi}F_{01}\big] - 2\lambda\alpha^2 A_1 = 0, \tag{42e}$$

which are identical to what we would have found directly from the field equations (36). The linear combination of the last two of these from which λ cancels gives

$$A_0\partial_t\big[\alpha^2 e^{-\phi}F_{01}\big] - A_1\partial_x\big[\alpha^2 e^{-\phi}F_{01}\big] = 0, \tag{43}$$

and we see that as expected, the $p_\mu A^\mu = 0$ characteristic has survived.

Although (43) results in $\alpha^2 e^{-\phi}F_{01} = \textit{constant}$, along these characteristics, it is difficult to solve the remaining equations. We will instead consider the simpler

class of solutions with $F_{01} = 0$, which also implies that $\lambda = 0$. In this case we find (writing the equation in both forms analogous to (25))

$$A_0 \partial_t A_1 - A_1 \partial_x A_1 = \tfrac{1}{2} l e^{\phi} \partial_x \phi, \tag{44a}$$

$$A_0 \partial_t A_0 - A_1 \partial_x A_0 = \tfrac{1}{2} l e^{\phi} \partial_t \phi. \tag{44b}$$

Both of α and ϕ will be functions of either $t + x$ or $t - x$, and for the the solutions ϕ that are either right- or left-moving ($\phi = \phi(t \mp x)$), we find (again assuming for forward-pointing $A_0 > 0$) that A_1 propagates in the same direction as ϕ. The equations (44) then become ($\phi' = \partial_t \phi(t \mp x)$):

$$\partial_t A_1 \mp \frac{A_1}{\sqrt{A_1^2 + l}} \partial_x A_1 = \tfrac{1}{2} l e^{\phi} \frac{\phi'}{\sqrt{A_1^2 + l}}, \tag{45a}$$

$$\partial_t A_0 \mp \frac{\sqrt{A_0^2 - l}}{A_0} \partial_x A_0 = \tfrac{1}{2} l e^{\phi} \frac{\phi'}{A_0}, \tag{45b}$$

and the gravitational wave ϕ is unaffected by the presence of the ether.

We have the following scenarios for the different types of ether: For a null ether all fields are decoupled, and all travel at the speed of light. For a timelike ether, the gravitational wave ϕ acts as a source in (45a), disturbing the ether. In this case disturbances in the ether travel slower than ϕ, and we generically expect shock formation in the ether. For a spacelike ether we have a similar situation, but (45b) tells us that the passing of the gravitational wave produces ether fluctuations that travel *faster* than light, and therefore faster than the gravitational wave disturbance itself. Similar to the flat spacetime case discussed in Section 4, the limit $A_0 \to 0$ does not allow a linearization that represents the behavior of A_0 as determined by the exact field equation (45b).

It would be interesting to explore more general solutions to (42) to find out how the back-reaction affects the propagation of gravitational waves. Although for the simple $F_{01} = 0$ solutions that we have considered it is a fairly simple matter to see that it is the passing of the gravitational wave that is disturbing the ether, in the more general case this is not obvious. Since this would probably require numerical analysis and since the speed of light becomes very large as $A_0 \to 0$, satisfying the Courant-Friedrichs-Lewy condition could be a delicate business.

8. Discussion and Conclusions

Although the nonlinear constraint imposed in these models is algebraic ($A^2 = l$ for some fixed l), we have shown that it nonetheless alters the causal evolution of the vector field. This is well-known when working in non-covariant gauges in Maxwell electrodynamics, where the resulting light cones should have no physical consequences. In the case at hand, the (co-)vector field A_μ is intended to represent a fundamental ether frame, giving a dynamically-determined, timelike, null or spacelike direction in spacetime. If this is the case, then it should not be observationally equivalent to a Maxwell field, the above constraint will have nontrivial consequences, and therefore the multiple light cones that we have shown exist in such models will be observable.

We have also shown that these models have a classical Hamiltonian that is not bounded from below. Although not necessarily a concern in classical conservative systems, we have to expect that any proposed quantization should seriously address the existence of a stable quantum vacuum. We have argued that things become

worse in this respect when terms like $(\nabla_\mu A^\mu)^2$ are introduced into the action, since then all four components of A_μ would have conjugate momenta, and we have even more freedom to find classical field configurations of arbitrarily large, negative energy. In addition, given the analysis of propagation speeds in linearized analyses of similar models [37], a simple light cone structure is not to be expected. Introducing curvature-coupling terms into the action cannot be expected to alleviate matters since the derivative-coupling that results is known to introduce altered propagation speeds for different fields [22], which has been explored in very similar vector field models in [27].

One of the main messages of this work is that it is important to realize that a linearized analysis does *not* necessarily tell the whole story (*i.e.*, a theory may have a 'linearization instability'). In the case of a spacelike ether we saw that it is quite possible to formally expand about a degenerate configuration, and although we know from a more complete analysis that there is a field with an infinite propagation velocity, this never appears in the linearized analysis. The canonical analysis has also shown that the Hamilton equations became singular at these solutions, and we therefore cannot expect that the resulting linearized equations to bear any resemblance to solutions of the exact field equations. This is precisely what was noticed in earlier work on vector fields coupled to gravity [27], and shows that the analysis in [18] is suspect. The same concerns may arise when considering the linearized analysis of vector models in [37], since the models are derivatively-coupled and the characteristics have not been determined.

In any case, the quantization of any of the models that we have discussed here is delicate, since the classical theory does not possess simple characteristics, fields in model are shock-forming, the classical energy is not bounded from below, and the model possesses linearization instabilities.

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References

[1] Andreas Albrecht and João Maguiejo. A time varying speed of light as a solution to cosmological puzzles. *Phys. Rev. D*, 59:043516, 1999. astro-ph/9811018.

[2] J. D. Barrow. Cosmologies with varying light speed. *Phys. Rev. D*, 59:043515, 1999. astro-ph/9811022.

[3] M. A. Clayton and J. W. Moffat. A scalar-tensor cosmological model with dynamical light velocity. gr-qc/0101126, 2001.

[4] John K. Webb, Victor V. Flambaum, Christopher W. Churchill, Michael J. Drinkwater, and John D. Barrow. A search for time variation of the fine structure constant. *Phys. Rev. Lett.*, 82:884–887, 1999.

[5] J. K. Webb, M. T. Murphy, V. V. Flambaum, V. A. Dzuba, J. D. Barrow, C. W. Churchill, J. X. Prochaska, and A. M. Wolfe. Further evidence for cosmological evolution of the fine structure constant. astro-ph/0012539, 2000.

[6] P. P. Avelino, C. J. Martins, G. Rocha, and P. Viana. Looking for a varying α in the cosmic microwave background. *Phys. Rev. D*, 62:123508, 2000. astro-ph/0008446.

[7] P. P. Avelino, C. J. A. P. Martins, and G. Rocha. Vsl theories and the doppler peak. *Phys. Lett. B*, 483:210, 2000.

[8] J. W. Moffat. Superluminary universe: A possible solution to the initial value problem in cosmology. *Int. J. Mod. Phys. D*, 2:351—365, 1993. gr-qc/9211020.

[9] Jerome Martin and Robert H. Brandenberger. The trans-planckian problem of inflationary cosmology. hep-th/0005209, 2000.

[10] J. Maguiejo. Stars and black holes in varying speed of light theories. *Phys.Rev. D*, 63:043502, 2001. `astro-ph/0010591`.

[11] M. A. Clayton and J. W. Moffat. Dynamical mechanism for varying light velocity as a solution to cosmological problems. *Phys. Lett. B*, 460:263–270, 1999. `astro-ph/9812481`.

[12] M. A. Clayton and J. W. Moffat. Scalar-tensor gravity theory for dynamical light velocity. *Phys. Lett. B*, 477:269–275, 2000. `astro-ph/9810112`.

[13] I. T. Drummond. Bi-metric gravity and “dark matter”. *Phys. Rev. D*, 63:043503, 2001. `astro-ph/0008234`.

[14] C. Armendáriz-Picón, T. Damour, and V. Mukhanov. k-inflation. *Phys.Lett. B*, 458:209–218, 1999. `hep-th/9904075`.

[15] C. Armendáriz-Picón, V. Mukhanov, and Paul J. Steinhardt. Essentials of k-essence. `astro-ph/0006373`, 2000.

[16] Ted Jacobson and David Mattingly. Gravity with a dynamical preferred frame. `gr-qc/0007031`, 2000.

[17] Ted Jacobson and David Mattingly. Generally covariant model of a scalar field with high frequency dispersion and the cosmological horizon problem. *Phys.Rev. D*, 63:041502, 2001. `hep-th/0009052`.

[18] V. Alan Kostelecký and Stuart Samuel. Gravitational phenomenology in higher-dimensional theories and strings. *Phys. Rev. D*, 40:1886–1903, 1989.

[19] E. Kiritis. Supergravity, D-brane probes and thermal super yang-mills: a comparison. *JHEP*, 9910:010, 1999. `hep-th/9906206`.

[20] Stephon H. S. Alexander. On the varying speed of light in a brane-induced FRW universe. *JHEP*, 0011:017, 2000. `hep-th/9912037`.

[21] C. Csaki, J. Erlich, and C. Grojean. Gravitational lorentz violations and adjustment of the cosmological constant in asymmetrically warped spacetimes. `hep-th/0012143`, 2000.

[22] Karel Kuchař. Geometrodynamics with tensor sources. IV. *J. Math. Phys.*, 18:1589–1597, 1977.

[23] H. A. Buchdahl. On the compatibility of relativistic wave equations in Riemann spaces. *Il Nuovo Cimento*, 25:486–496, 1962.

[24] H. A. Cohen. Covariant wave equations for charged particles of higher spin in an arbitrary gravitational field. *Il Nuovo Cimento*, 52A:1242–1252, 1967.

[25] Giorgio Velo and Daniel Zwanzinger. Noncausality and other defects of interaction Lagrangians for particles with spin one and higher. *Phys. Rev.*, 188:2218–2222, 1969.

[26] C. Aragone and S. Deser. Constraints on gravitationally coupled tensor fields. *Il Nuovo Cimento*, 3A:709–720, 1971.

[27] J. A. Isenberg and J. M. Nester. The effect of a gravitational interaction on classical fields: A Hamiltonian–Dirac analysis. *Ann. Phys. (NY)*, 107:56–81, 1977.

[28] Clifford M. Will and Jr. Nordtvedt, Kenneth. Conservation laws and preferred frames in relativistic gravity. I. Preferred-frame theories and an extended PPN formalism. *Astrop. J.*, 177:757–7774, 1972.

[29] Clifford M. Will and Jr. Nordtvedt, Kenneth. Conservation laws and preferred frames in relativistic gravity. II. Experimental evidence to rule out preferred-frame theories of gravity. *Astrop. J.*, 177:775–792, 1972.

[30] Michael Renardy and Robert C. Rogers. *An Introduction to Partial Differential Equations*, volume 13 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1992.

[31] Y. Choquet–Bruhat, C. DeWitt–Morette, and M. Dillard–Bleik. *Analysis, Manifolds and Physics*, volume 1. North Holland, New York, 1989.

[32] Robert M. Wald. *General Relativity*. The University of Chicago Press, Chicago, 1984.

[33] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space–time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, New York, 1973.

[34] D. Brill and S. Deser. Instability of closed spaces in general relativity. *Comm. Math. Phys.*, 32:291–304, 1973.

[35] M. A. Clayton. Linearisation instabilities of the massive nonsymmetric gravitational theory. *Class. Quantum Grav.*, 13:2851–2864, 1996.

[36] Alexandre J. Chorin and Jerrold E. Marsden. *A Mathematical introduction to Fluid Mechanics*, volume 4 of *Texts in Applied Mathematics*. Springer-Verlag, New York, third edition, 1992.

[37] Clifford M. Will. *Theory and experiment in gravitational physics*. Cambridge University Press, New York, revised edition, 1993.

Physics Department, Acadia University, Wolfville, Nova Scotia B0P 1X0 Canada.
E-mail address: michael.clayton@acadiau.ca